SOME APPROACHES TOWARDS CONSTRUCTING OPTIMALLY EFFICIENT MULTIGRID SOLVERS FOR THE INVISCID FLOW EQUATIONS
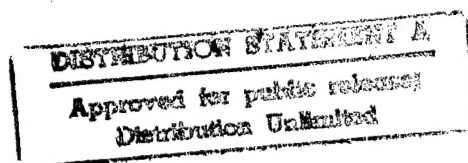
David Sidilkover

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This paper summarizes briefly these developments and outlines the
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Some approaches towards constructing optimally efficient multigrid solvers for the inviscid flow equations*

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Abstract

Some important advances took place during the last several years in the development of genuinely multidimensional upwind schemes for the compressible Euler equations. In particular, a robust, high-resolution genuinely multidimensional scheme which can be used for any of the flow regimes computations was constructed (see [1–3]). This paper summarizes briefly these developments and outlines the *fundamental* advantages of this approach.

1 INTRODUCTION

The efficiency of the existing steady-state multigrid solvers routinely used for the flow problems in engineering practice is still very poor. There clearly exists a pressing need for more efficient algorithms.

In order to obtain a truly efficient steady-state solver, some *fundamental* issues concerning different aspects of an algorithm need to be addressed. The recently proposed genuinely multidimensional approach towards the construction of the discrete schemes for the compressible flow resolves some of these issues. A discussion in this regard is the main subject of this paper.

1.1 Genuinely multidimensional schemes

The quest for a genuinely multidimensional upwind scheme began more than a decade ago. Initially it was motivated chiefly by the expectation that, once such a scheme is developed, it will imitate the physics of the fluid flow more accurately than the standard dimension-by-dimension approach. It was, however, suggested later in [4],[5] that improving the efficiency of the the steady-state solvers may be the most important reason for developing the genuinely multidimensional approach.

One of the main difficulties in the numerical treatment of compressible flow is the possible presence of shocks in the solution. It is well known that a scheme that is both second order accurate and avoids under- and overshoots (which may trigger a nonlinear instability) near discontinuities has to be nonlinear. Such a scheme has to incorporate the so-called *high-resolution* mechanism, i.e. a smoothness monitor, that is usually implemented in the form of a flux-limiter. Initially, such schemes were developed for the one-dimensional case. Then, extending this approach to multidimensions was done on a dimension-by-dimension basis. The well known fact, however, is that the Gauss-Seidel relaxation is unstable when applied in conjunction with such schemes [6]. Therefore, the standard multigrid solvers have to resort to the defect-correction technique or multistage Runge-Kutta relaxation and the efficiency of such solvers may be poor. A closer look reveals that the standard high-resolution discretizations suffer from the following deficiency: the high-frequency error components may be (nearly) invisible to the residuals of the discrete equations, i.e. the discrete scheme is (nearly) unstable. In turn this means that it may be inherently impossible to construct a good smoother (an important ingredient of a multigrid solver) using these discrete schemes.

A genuinely multidimensional advection scheme was constructed in [4,5]. The scheme was named "genuinely multidimensional" since it imitates well the

anisotropy of the advection phenomena in two dimensions: artificial dissipation is added only along the streamline, while the high-resolution mechanism affects significantly the cross-flow direction only. The key feature of this scheme is the two-dimensional limiter, i.e. the argument of a limiter-function is the ratio of finite differences in two different coordinate directions. The scheme was formulated in the control-volume context for Cartesian grids and relied on the compact 9-point-box stencil. The *fundamental* advantage of this approach is that the two-dimensional high-resolution mechanism does not damage the stability properties of the discretization.

The so-called "residual distribution" (or "fluctuation-splitting") schemes for scalar advection equation on unstructured triangular grids were presented in [7]. It was found later that these schemes have some links to the aforementioned genuinely two-dimensional control-volume approach for the advection equation. Exploration of these links led to the unification of the two approaches and resulted in a scheme that incorporated two-dimensional limiters and was formulated for unstructured triangular grids. This scheme (like that presented in [4],[5]) can be given a purely *algebraic* interpretation. However, the task of extending these ideas to systems of equations appeared to be a complicated one.

Consider a hyperbolic system of partial differential equations in two dimensions

$$\mathbf{u}_t + A\mathbf{u}_x + B\mathbf{u}_y = 0, \tag{1}$$

where $\mathbf{u}$ is the vector of size N and A, B are $N \times N$ matrices. The matrices A and B in general do not commute. This means that they cannot be diagonalized simultaneously, i.e. the system cannot be written as N advection equations.

A prolonged effort was to represent locally the physics of compressible flow by finite number of simple waves using the local gradients of the solution (in the spirit of [8]) with intention to apply a genuinely two-dimensional advection scheme to each one of the simple waves. However, the schemes constructed in this way for the Euler equations suffered from a severe lack of robustness.

The breakthrough approach that resulted in a robust genuinely multidimensional scheme, suitable for the computations of the entire range of the flow regimes was presented in [1]. Then it was described in more detail including the implications for multigrid and extension to 3D in [2] and [3]. The key idea was not to try to apply the multidimensional advection schemes to systems, but rather the same strategy that was used to construct the scalar scheme. The *algebraic* interpretation of the advection scheme played an important role at this point. It was crucial to recognize that a certain linear first order scheme based

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on standard upwind methodology can be used as a basic building block for the hyperbolic systems as well as for the scalar advection. The multidimensional high-resolution corrections are then applied in a formal way similarly to the scalar case. The resulting scheme for the Euler equations was demonstrated to produce a very good quality solution for subsonic, transonic and supersonic regimes. The approach was called "genuinely-multidimensional" since it can be argued that it leads to a discrete scheme whose artificial dissipation is a rotationally-invariant differential operator (in other words – the artificial dissipation operator is independent to a certain extent of the grid direction). It was not clear if this particular property is of any direct practical importance.

The constructed high-resolution scheme for the Euler equations relies on a compact stencil. The result of this property is a smaller error in smooth regions and better resolution of discontinuities, comparing to the standard dimension-by-dimension approach. However, in our view, these are only *marginal* improvements. The *fundamental* advantage of this approach is that it leads to a scheme that combines high-resolution and good stability properties. It was demonstrated in [2,3] that the Collective Gauss-Seidel relaxation is stable when applied directly to the resulting high-resolution discretization of the hyperbolic systems. This results in a very simple, efficient and robust multigrid solver for the compressible Euler equations, suitable for the entire range of flow regimes.

Some researchers who were previously pursuing other directions adopted the genuinely-multidimensional approach proposed in [1–3] and attributed to it a term "Positive Matrix Distribution Schemes". A modification of the underlying first order scheme (the system N-scheme) aimed at improving the discontinuity resolution was proposed by van der Weide and Deconinck in [9].

It should be mentioned that important steps towards the construction of a genuinely multidimensional schemes for the Euler equations were made by Colella [10,11], LeVeque [12] and Radvogin [13]. However, the nonlinear high-resolution corrections in these schemes rely on one-dimensional limiters, which introduces some of the dimension-by-dimension flavor.

Another very interesting approach was proposed in [14]. A discretization for the triangular unstructured grids for the Euler equations was developed. The problem was that the scheme was linear, i.e. it did not incorporate any nonlinear high-resolution mechanism.

1.2 Multigrid for advection dominated problems

One of the major reasons for the poor efficiency of the standard flow solvers (see [15]) is the fact that for advection dominated problems the coarse grid

provides only a fraction of the needed correction for certain error components. It is well known that the steady Euler equations contain two different factors: the advection and the Full-Potential type operators. The latter is either of elliptic or hyperbolic type depending on the flow regime (subsonic or supersonic). The difficulty mentioned above can be avoided ([15]) by constructing a solver that distinguishes between different factors of the system and treats each one appropriately. In the subsonic case, for instance, the advection factor can be treated by marching and the elliptic factor by multigrid. The efficiency of such an algorithm will be essentially the same as that of the multigrid solver for the elliptic part only. Such algorithms are referred to as "essentially optimal". An approach to achieve a separation between the co-factors – the so-called Distributive Gauss-Seidel relaxation – was proposed in [15]. It was demonstrated in [16] that using this approach one can obtain the essentially optimal multigrid efficiency for a staggered-grid discretization of the incompressible Navier-Stokes equations. It is interesting to mention that a similar observation was made earlier in [17].

A related approach was proposed by Ta'asan [18] for the incompressible and compressible subsonic Euler equations. The staggered-grid discretization is based on the *canonical* variables formulation (see [19]), that expresses the partitioning of the steady Euler equation into elliptic and advection factors. The essentially optimal multigrid efficiency was demonstrated using this approach for subsonic flow and body-fitted grids. A possible limitation of this approach may be that it is not directly generalizable for the viscous flow.

Another way towards achieving the optimal multigrid efficiency is based on the pressure equation formulation of the Euler equations. We describe it briefly in this paper as well. This approach is based on a very old idea and is a generalization [20]. Its main virtue is simplicity. It can also be classified as Weighted Gauss-Seidel relaxation [15]. An extensive set of numerical computations using this scheme together with more details regarding the implementaion is reported in [21]). The limitation of this approach, however, is that it is not clear so far if it is generalizable to *viscous* compressible flow.

Following the work of Ta'asan, some researchers attempted to apply the idea of partitioning the Euler equations towards the construction of discrete schemes (see [22] and [23]). It is well-known that the two-dimensional Euler equations in supersonic case can be written as four locally decoupled advection equations (see [24]). This property was used as a basis for applying the advection schemes to discretize the system in this case. In subsonic case, however, the distinction was made between the advection and the elliptic ("acoustic subsystem") partitions. The treatment of transonic flow, however, was problematic since it required matching of two different discretizations accross the sonic lines. Another drawback of these approaches was that they are cannot be generalized to three dimensions (see [9]). To conclude, the discrete schemes constructed in

this way suffered from a lack of robustness and generality. No optimal multi-grid efficiency was demonstrated either. Finally, some of the researchers who previously followed this direction adopted the genuinely-multidimensional approach proposed in [1–3] (see [9]).

1.3 What this paper is about

In this paper we first present a brief review of the construction of genuinely multidimensional schemes for the scalar advection and the compressible Euler equations. We summarize the basic properties of the discretizations, emphasizing those that are unique to this approach and are of *fundamental* importance for practical purposes.

The separation of the co-factors can be in general achieved in two ways. One way is to cast equations into such a form that the different co-factors can be discretized separately. The canonical variables approach by Ta'asan [19] can be classified as such. The pressure equation based scheme, presented briefly in this paper, belongs to this category as well. The key advantage of this type of methods is in their simplicity (this is especially true for the pressure equation based scheme). The disadvantage, though, may be in the unsufficient generality. Another more general way to achieve the optimal multigrid efficiency (see [15]) is to discretize the equations in some primitive form and to apply a relaxation of the Distributive Gauss-Seidel type. Such relaxation should be designed in such a way that it distinguishes between the different co-factors of the system and treats each one of them appropriately. It appears, however, that in order to achieve this not any discrete schemes are suitable, but only those satisfying a certain condition. As it was mentioned before, it is not clear, what are the direct practical implications of the *genuine multidimensionality* (or the rotational invariance of the artificial dissipation) property of the approach. However, we present in this paper a heuristc argument suggesting that the *genuine multidimensionality* is closely related to the *factorizability* property of the discretization. The latter is of *fundamental* practical importance. It is necessary in order to construct a Distributive Gauss-Seidel relaxation that will allow to decouple the advection and Full-Potential co-factors of the Euler system and thus to obtain an optimally efficient multigrid solver. This approach is very general since it does not require to cast the equations into any special form.
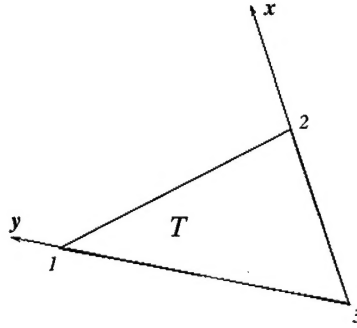
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Fig. 1. Triangle.

2 MULTIDIMENSIONAL UPWINDING

In this section we briefly review the construction of the high-resolution genuinely multidimensional upwind schemes for the scalar advection equation and for the Euler system. Consider a general triangular grid covering the domain. Assume the discrete unknowns are defined at the grid nodes thus defining a linear function and allowing us to evaluate the gradients of the current solution approximation on each triangle. The approach is to construct discrete equations which are to be solved at each node from the portions of the residuals of the equations computed on the triangles having this node as a common vertex. In other words, portions of the residual of the equation evaluated on a particular triangle contribute to the construction of the discrete equations at the nodes of this triangle. The problem is to find the exact rules for this construction, so that the resulting discrete equations will have certain desirable properties.

2.1 Advection scheme

We consider triangular element T, and choose two out of the three faces. Then we write the advection equation we wish to solve in the local coordinate system aligned with these two faces

$$u_t + au_x + bu_y = 0 \tag{2}$$

Without loss of generality we can consider a linear constant coefficient equation, since in general non-linear case we can linearize the equation on each triangle ([25]).

We can write the discrete equation at the grid node i in the following form

$$c_{i,i}u_i - \sum_{j \neq i} c_{i,j}u_j = 0, \tag{3}$$

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Definition 1 *The discrete scheme* (3) *is said to be of the positive type if* $c_{i,j} \geq 0$ *for all* j.

Numerical soltuions obtained using a positive scheme satisfy a discrete *maximimum principle*. This property is useful to ensure that the discrete solution will be non-oscillatory near discontinuities.

We shall outline here the construction of a positive advection scheme. Residual of the equation (2) can be represented as a sum of two portions

$$r = r^x + r^y, \tag{4}$$

where

$$r^x = -S_T a u_x^h; \quad r^y = -S_T b u_y^h. \tag{5}$$

Residual of the equation (2) on the triangle T contributes to the construction of the discrete equations to be solved at each of the three nodes of the triangle according to the following residual distribution formulae

$$\begin{aligned}
\text{node } 1 &\leftarrow r^x(1 - \text{sign}(a)) \\
\text{node } 2 &\leftarrow r^x(1 + \text{sign}(a)) + r^y(1 - \text{sign}(b)) \\
\text{node } 3 &\leftarrow r^y(1 + \text{sign}(b))
\end{aligned} \tag{6}$$

It easy to see that this construction results in a positive scheme since for any real number z we have the following inequality $\pm z(1 \pm \text{sign}(z)) \geq 0$. The accuracy of such a scheme, though, is only first order.

Definition 2 *A discrete scheme is called* linearity preserving *if whenever the residual* r *on the triangle* T *vanishes, the contributions due to this residual lead to a zero update of the solution at each of the three nodes of the triangle.*

A *linearity preserving* scheme is second order accurate.

Define the following quantities

$$r^{x^*} = r^x + r^y \Psi(q); \quad r^{y^*} = r^y + r^x \Psi(q)/q \tag{7}$$

where $q = -r^x/r^y$. In this paper we assume that Ψ is the *minmod* limiter. Substituting r^{x^*}, r^{y^*} instead of r^x, r^y into (6) we obtain a high-resolution scheme. The important feature here is the two-dimensionality of the limiter, i.e. the fact that the argument of the limiter-function is a ratio of numerical derivatives in two different coordinate direction ([5],[4]).

Using the following limiter identity

$$r^y \Psi(q) \equiv -r^x \Psi(q)/q,$$ (8)

it is easy to verify that the constructed nonlinear scheme is indeed both *positive* and *linearity preserving*.

2.2 *Extension to the Euler system*

Consider a hyperbolic system of partial differential equations

$$\mathbf{u}_t + A\mathbf{u}_x + B\mathbf{u}_y = 0.$$ (9)

The discrete equation approximating (9) at node i can be written as follows

$$C_{i,i}\mathbf{u}_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} C_{i,j}\mathbf{u}_j = 0,$$ (10)

Property of positivity can be formally extended to the system case.

Definition 3 *The discrete scheme* (10) *is said to be of the positive type if the matrices $C_{i,j}$ (for all j) have non-negative eigenvalues.*

It is not clear, however, how to generalize the *maximum principle* for systems. No conclusions can be derived from the positivity property unless the additional assumption about the symmetry of the matrices $C_{i,j}$ is made. The energy stability property of the scheme can be demonstrated in this case. However, for the Euler system, this would require the use of the *symmetrizing* variables formulation, which is *non-conservative*. This makes the energy stability property too restrictive to be of substantial practical importance for the Euler equations.

It is interesting to note though that the standard high-resolution schemes, if carefully implemented, are of the *positive* type. Therefore, we aimed at constructing a genuinely multidimensional high-resolution scheme ([1–3]), which is of the *positive* type as well.

Assume that the hyperbolic system of equations (9) is written in the non-orthogonal coordinate frame aligned with the two of the faces of triangle T (Fig.1). Residuals of the system on triangle T can be represented as a sum of two portions

$$\mathbf{R} = \mathbf{R}^x + \mathbf{R}^y,$$ (11)

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where

$$\mathbf{R}^x = -S_T A \mathbf{u}^h_x;$$
$$\mathbf{R}^y = -S_T B \mathbf{u}^h_y. \qquad (12)$$

Consider the following residual distribution formula

$$\text{node } 1 \leftarrow \mathbf{R}^x (I - \text{sign}(A))$$
$$\text{node } 2 \leftarrow \mathbf{R}^x (I + \text{sign}(A)) + \mathbf{R}^y (I - \text{sign}(B)) \qquad (13)$$
$$\text{node } 3 \leftarrow \mathbf{R}^y (I + \text{sign}(B))$$

Assuming that matrix M has a complete set of real eigenvalues (definition of the hyperbolicity of a system) it is easy to see that matrix $\pm M(I \pm \text{sign}(M))$ is non-negative definite. This means that the scheme defined by (13) is of the positive type by construction (as an upwind scheme is expected to be).

In order to obtain a *positive* high-resolution genuinely multidimensional scheme for a hyperbolic system we may have first to rewrite system (9) (as it was done in [1–3]) in a different set of variables. The *auxiliary* variables $\mathbf{v} = (s, u, v, p)^T$ (see [1–3]) are a good choice for the Euler system (here s is the entropy, u, v - the velocity components and p is the pressure). System (9) rewritten in varibales $\mathbf{v}$ takes the following form

$$\mathbf{v}_t + \mathcal{A}\mathbf{v}_x + \mathcal{B}\mathbf{v}_y = 0 \qquad (14)$$

where

$$\mathbf{u} = T\mathbf{v} \qquad (15)$$

As before, we can compute residual $\mathbf{r}$ of system (14) on triangle T and represent it as a sum of two portions:

$$\mathbf{r} = \mathbf{r}^x + \mathbf{r}^y, \qquad (16)$$

where

$$\mathbf{r}^x = -S_T \mathcal{A}\mathbf{v}^h_x;$$
$$\mathbf{r}^y = -S_T \mathcal{B}\mathbf{v}^h_y. \qquad (17)$$

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Considering the following residual distribution formula

$$\text{node } 1 \leftarrow T\mathbf{r}^x(I - \text{sign}(\mathcal{A}))$$
$$\text{node } 2 \leftarrow T[\mathbf{r}^x(I + \text{sign}(\mathcal{A})) + \mathbf{r}^y(I - \text{sign}(\mathcal{B}))] \quad (18)$$
$$\text{node } 3 \leftarrow T\mathbf{r}^y(I + \text{sign}(\mathcal{B})),$$

we arrive at a positive first order accurate scheme that is *identical* to (13).

Introduce the following quantities

$$r_i^{x^*} = r_i^x + r_i^y\Psi(q_i); \quad r_i^{y^*} = r_i^y + r_i^x\Psi(q_i)/q_i \quad (19)$$

with $q_i = -r_i^x/r_i^y$, and $i = 1, ..., N$ Denote by $\mathbf{r}^{x^*}$ and $\mathbf{r}^{y^*}$ vectors whose components are $r_i^{x^*}$ and $r_i^{y^*}$ ($i = 1, \ldots, N$) respectively. The high-resolution genuinely two-dimensional scheme can be obtained by substituting $\mathbf{r}^{x^*}$ and $\mathbf{r}^{y^*}$ instead of the $\mathbf{r}^x$ and $\mathbf{r}^y$ into (18).

Using the limiter identity

$$r_i^y\Psi(q_i) \equiv -r_i^x\Psi(q_i)/q_i, \text{ for } i = 1, ..., N \quad (20)$$

it is possible to show that the genuinely two-dimensional high-resolution scheme is both positive and linearity preserving. We emphasize here again, that in order to achieve this property for the Euler equations, it was necessary to use another (non-conservative) form of the equations when introducing the high-resolution mechanism ([1–3]). The *auxiliary* variables formulation was suitable for this purpose. The only justification for the desirability of the positivity property for systems of equations is that the standard high-resolution schemes, if properly implemented, have this property.

Remark 4 *Recall that when constructing a discrete scheme we have chosen arbitrarily two out of three faces of the trinagle T (see Fig.1) for the purpose of numerical derivatives evaluation. Note, that any choice will result in a scheme with good properties. In the case of the scalar advection, though, if we use* **two faces which are both of the same kind with respect to the flow direction (inflow or outflow), the resulting scheme will rely on the narrowest possible stencil. Therefore, it will have a smaller cross-stream error coefficient and will** *provide a somewhat better disontinuity resolution. It is possible in the case of the Euler system to optimize the resolution of a specific sharp layer (shock, contact discontinuity or shear layer) by the appropriate choice of the faces (two "closest" to the layer direction. In general, though, it seems reasonable* **to choose two faces which are either inflow or outflow. This will provide better**

resolution of the contact discontinuities and shear layer, while shocks are well resolved anyway.

In addition to the smaller error in smooth regions and sharper resolution of discontinuities (compared to the standard dimension-by-dimension methods), the constructed scheme also offers a possibility to optimize the stencil in order to resolve a particular discontinuity layer (by choosing two faces of a triangular element that are the closest to the direction of this layer). The important advantage of this approach, however, is that the genuinely multidimensional approach presented here leads to a scheme which has both high-resolution and good stability properties.

3   SEPARATION OF CO-FACTORS

Incompressible steady Euler equations in two dimensions can be written in the following matrix form (assuming that the fluid density $\rho \equiv 1$)

$$Lu = 0, \tag{21}$$

where

$$\mathbf{u} = (u, v, p)^T, \tag{22}$$

$$L = \begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}, \tag{23}$$

and $Q = \vec{U} \cdot \nabla$ is the advection operator. In order to find out what is the type the system of equations we can look at the determinant of matrix L

$$det(L) = -Q^2 \Delta \tag{24}$$

The co-factors of the determinant are of the advection and another of the elliptic types.

Compressible steady Euler equations in two dimensions can be written in the following matrix form

$$Lu = 0, \tag{25}$$

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where

$$\mathbf{u} = (s, u, v, p)^T, \qquad (26)$$

$$L = \begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & \rho Q & 0 & \partial_x \\ 0 & 0 & \rho Q & \partial_y \\ 0 & \rho\partial_x & \rho\partial_y & \frac{1}{c^2}Q \end{pmatrix}. \qquad (27)$$

The determinant of matrix L in this case

$$det(L) = \rho^2 (Q)^2 [\frac{1}{c^2}Q^2 - \Delta] \qquad (28)$$

The determinant has two distinct co-factors: one of advection and another of the Full-Potential type.

In the rest of this paper we shall consider the subsonic regime only. The Full-Potential factor in this case is of the elliptic type.

It was suggested in [15] that the different co-factors should also be treated differently (each one in the appropriate way). One way to do this is to cast the equations into such a form that the co-factors can be discretized separately. Canonical variables formulation [18] as well as the pressure equation based schemes, that are described in the next section, belong to this category. A more general approach (as proposed by Brandt in [15]) is to discretize the equations in some primitive form, but to design relaxation of Distributive Gauss-Seidel type (DGS) such that it will separate the treatment of co-factors. However, the discrete scheme suitable for this purpose should be *factorizable*, i.e. satisfy the discrete analog of the property (28). We show in this paper that the genuinely multidimensional upwind approach leads to a *factorizable* scheme.

4 PRESSURE EQUATION APPROACH

In this section we briefly describe the pressure equation based approach first for incompressible and then for compressible cases.

4.1 Incompressible case

Considering a triangular (unstructured) grid and assuming that the unknowns are located at grid vertices, we can define piecewize-linear functions approximating each unknown. Residual of the Euler equations can then be evaluated on each triangular element of the grid:

$$\mathbf{r} = L^h \mathbf{u}^h, \tag{29}$$

where $\mathbf{u}^h = (u^h, v^h, p^h)^T$,

$$L^h = \begin{pmatrix} Q^h & 0 & \partial_x^h \\ 0 & Q^h & \partial_y^h \\ \partial_x^h & \partial_y^h & 0 \end{pmatrix}, \tag{30}$$

and

$$Q^h \equiv u^h \partial_x^h + v^h \partial_y^h \tag{31}$$

is the discrete advection operator.

We would like then to construct the discrete equations to be solved at a certain grid node by assembling in a certain way the residuals of the Euler equations computed on the triangular element having this node as a common vertex. These equations can be written in the following form

$$P^h \mathbf{r} \equiv P^h L^h \mathbf{u}^h = 0 \tag{32}$$

In order to obtain the equations with desirable properties, i.e. to achieve the decoupling of the co-factors, we can define the following assembly matrix P^h

$$P^h = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \partial_x^h & \partial_y^h & (Q^*)^h \end{pmatrix}, \tag{33}$$

where

$$(Q^*)^h(f^h) \equiv -\partial_x^h(u^h f^h) - \partial_y^h(v^h f^h) \tag{34}$$

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is the operator adjoint to Q^h. Then

$$P^h \mathbf{r} = \begin{pmatrix} Q^h & 0 & \partial_x^h \\ 0 & Q^h & \partial_y^h \\ 0 & 0 & \Delta^h \end{pmatrix} \begin{pmatrix} u^h \\ v^h \\ p^h \end{pmatrix} + s.p.t. \tag{35}$$

where Δ^h is the discrete Laplacian and "s.p.t." stands for subprincipal terms which appear due to the non-constancy of the advection coefficients and are not important for the purpose of constructing a relaxation scheme [15]. The maxtrix of the finite difference operators in (35) is upper-triangular. The pressure is subject to a Poisson equation wich is decoupled (upto subprincipal terms) from the rest of the system. The standard Gauss-Seidel relaxation can, therefore, be applied to it as a smoother. The momentum equations can be looked at as advection equations with (known) forcing terms (pressure derivatives). The ideas of multidimensional upwinding can be applied to discretize them. The strategy applied to relax the system can be then to relax the pressure first and then to update the velocities components by relaxing the momentum equations in the downstream direction. An extensive set of computational results using this approach is presented in [21].

4.2 Compressible case

Similarly to the incompressible case, residuals of the equations can be evaluated on each triangular element:

$$\mathbf{r} = L^h \mathbf{u}^h, \tag{36}$$

where $\mathbf{u}^h = (s^h, u^h, v^h, p^h)^T$ and

$$L^h = \begin{pmatrix} Q^h & 0 & 0 & 0 \\ 0 & \rho^h Q^h & 0 & \partial_x^h \\ 0 & 0 & \rho^h Q^h & \partial_y^h \\ 0 & \rho^h \partial_x^h & \rho^h \partial_y^h & \frac{1}{(c^h)^2} Q^h \end{pmatrix} \tag{37}$$

and c^h is the (discrete) speed of sound. Again, we construct the discrete equations to be solved at each node assembling in a certain way the residuals of the equations on the elements surrounding this node

$$P^h \mathbf{r} \equiv P^h L^h \mathbf{u}^h = 0 \tag{38}$$

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Choosing the assembly matrix to be

$$
P^h = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & \partial_x^h & \partial_y^h & (Q^*)^h \end{pmatrix},
\tag{39}
$$

we obtain

$$
P^h \mathbf{r} = \begin{pmatrix} Q^h & 0 & 0 & 0 \\ 0 & \rho^h Q^h & 0 & \partial_x^h \\ 0 & 0 & \rho^h Q^h & \partial_y^h \\ 0 & 0 & 0 & \Delta^h - \frac{1}{(c^h)^2}(Q^h)^2 \end{pmatrix} \begin{pmatrix} s^h \\ u^h \\ v^h \\ p^h \end{pmatrix} + s.p.t.
\tag{40}
$$

The principal part of last equation is the discrete Prandtl-Glauert (or Full Potential) operator acting on the pressure. This operator is of the elliptic type in the subsonic regime. The solution process of the resulting discrete equations is very similar to that for the incompressible case. Some numerical tests illustrating the efficiency of the multigrid solver based on this discretization are presented in the §6.2.

5 UPWINDING AND CO-FACTORS SEPARATION

Now we return to the genuinely multidimensional upwind approach. We would like to construct a linear first order upwind "positive" scheme such that it is factorizable and is also upgradable to second order using the genuinely two-dimensional high-resolution mechanism.

First, we shall take a closer look at the one-dimensional case.

5.1 One-dimensional case

Consider a first order upwind scheme for the one-dimensional Euler equations. Without loss of generality we consider the primitive variable formulation

$$
L^h \mathbf{u}^h = 0
\tag{41}
$$

where

$$\mathbf{u}^h = (s^h, u^h, p^h)^T \tag{42}$$

and

$$L^h = \begin{pmatrix} -\frac{h}{2}|u|\partial_{xx}^h + Q^{2h} & 0 & 0 \\ 0 & \rho(-\frac{h}{2}c\partial_{xx}^h + Q^{2h}) & \partial_x^{2h} - \frac{h}{2}\frac{u}{c}\partial_{xx}^h \\ 0 & \rho(\partial_x^{2h} - \frac{h}{2}\frac{u}{c}\partial_{xx}^h) & -\frac{h}{2c}\partial_{xx} + \frac{1}{c^2}Q^{2h} \end{pmatrix}, \tag{43}$$

where h is a meshsize, ∂_{xx}^h is a central approximation of the second derivative, ∂_x^{2h} is a central approximation of the first derivative and $Q^{2h} = u\partial_x^{2h}$ is the advection operator.

5.1.1 Factorization

The determinant of L^h:

$$det(L^h) = \rho(-\frac{h}{2}|u|\partial_{xx}^h + Q^{2h})[(1 - M^2)\partial_{xx}^h] \tag{44}$$

The first factor is the upwind scheme approximating an advection operator corresponding to the entropy equations. The Full-Potential factor is approximated by a "short" central difference. The issue of factorization appears to be trivial in this case, since the momentum and the pressure equations correspond solely to the elliptic factor.

5.1.2 Distributive Gauss-Seidel relaxation

Introducing new variables

$$(s^h, u^h, p^h)^T = \mathbf{u}^h = M^h\mathbf{w}^h = M^h(s^h, w^h, \phi^h)^T \tag{45}$$

the Euler system will take the following form
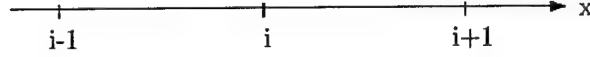
$$L^h M^h \mathbf{w}^h = 0 \tag{46}$$

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Fig. 2. One-dimensional grid

Assuming that that

$$
M^h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}\frac{1}{c}\partial^h_{xx} + \frac{1}{c^2}Q^{2h} & \partial^{2h}_x - \frac{h}{2}\frac{u}{c}\partial^h_{xx} \\ 0 & -\rho(\partial^{2h}_x - \frac{h}{2}\frac{u}{c}\partial^h_{xx}) & \rho(\frac{h}{2}c\partial^h_{xx} - Q^{2h}) \end{pmatrix}
\tag{47}
$$

we obtain

$$
L^h M^h = \begin{pmatrix} -\frac{h}{2}|u|\partial^h_{xx} + Q^{2h} & 0 & 0 \\ 0 & \rho(1 - M^2)\partial^h_{xx} & 0 \\ 0 & 0 & \rho(1 - M^2)\partial^h_{xx} \end{pmatrix}
\tag{48}
$$

The philosophy of the Distributive Gauss-Seidel relaxation is as follows: We would like to store at the gridpoints the primitive variables **u** and not the auxiliary variables **w**. For this purpose the updates in w and ϕ corresponding to the relaxing the elliptic factor at point i should be translated into the updates in u, p at points $(i-1), (i), (i+1)$ according to M^h.

5.1.3 DGS relaxation and the Riemann solver

It is well known that the one-dimensional Euler system can be diagonalized, i.e. rewritten as a set of (locally) decoupled advection equations for the *characteristic* variables $(s, \alpha^+, \alpha^-)^T$, where

$$
\alpha^+_x = \rho c u_x + p_x, \quad \text{and} \quad \alpha^-_x = \rho c u_x - p_x.
\tag{49}
$$

Some algebra reveals that relaxing the Full-Potential (elliptic) factor corresponds to:

- at point $(i-1)$ - update α^+, keep α^- the same;
- at point $(i+1)$ - update α^-, keep α^+ the same;
- at point i - update both α^+ and α^- .

We can conclude that there are some links between the characteristic variables formulation (approximate Riemann solver) and the design of the DGS.

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5.2 Two-dimensional case

We shall look now for a *factorizable* upwind scheme in two dimensions.

5.2.1 Dimension-by-dimension approach

Considering here the isentropic case (no loss of generality for the purpose of the discussion presented here), we can write such a scheme in the following symbolic form

$$L^h \mathbf{u}^h = 0 \tag{50}$$

The modified equations (or the First Differential Approximation – FDA) corresponding to the scheme dimension-by-dimension

$$FDA(L^h) =$$
$$\begin{pmatrix} \rho[-\frac{h}{2}(c\partial_{xx} + |v|\partial_{yy}) + Q] & 0 & \partial_x - \frac{h}{2}\frac{1}{c}u\partial_{xx} \\ 0 & \rho[-\frac{h}{2}(|u|\partial_{xx} + c\partial_{yy}) + Q] & \partial_y - \frac{h}{2}\frac{1}{c}v\partial_{yy} \\ \rho(\partial_x - \frac{h}{2}\frac{u}{c}\partial_{xx}) & \rho(\partial_y - \frac{h}{2}\frac{v}{c}\partial_{yy}) & -\frac{h}{2}\frac{1}{c}\Delta + Q \end{pmatrix} \tag{51}$$

It is easy to see that the matrix (51) cannot be factorized.

5.2.2 Genuinely multidimensional approach

The approach towards the construction of discrete schemes for the Euler equations ([1,2]) was called "genuinely multidimensional" since it leads to schemes that retain (to a certain extent) the rotational invariance property of the Euler equations. Namely, it can be argued that the artificial dissipation terms present in these schemes approximate a rotationally invariant differential operator. In its turn this may mean that the waves oblique to the grid are "properly upwinded" or, in other words, the same approximate Riemann solver can be "recovered" in an arbitrary direction.

It is not clear whether or not this property, though intuitively appealing, is of any direct practical importance for the steady-state computations. Therefore, we do not discuss it in detail. However, we have observed previously that there are some links between the approximate Riemann solver and the design of DGS in one dimensional case. Therefore, the following conjecture seems reasonable.

Conjecture *A genuinely multidimensional scheme is factorizable.*

It was pointed out in [2] that some of the multidimensional second order corrections can be added without limiters resulting in a linear 'positive' scheme with essential multidimensional character. Namely, for the u-momentum equation in subsonic case, those are the cross-derivative correction terms that compensate for the loss of accuracy due to the artificial dissipation in x direction. For the v-momentum equations those will be the terms that compensate for the loss of accuracy due to the artificial dissipation in y-direction.

Writing such a scheme in the symbolic form

$$L^h_{[2D]}\mathbf{u}^h = 0, \tag{52}$$

and considering the corresponding FDA

$$FDA(L^h_{[2D]}) =$$
$$\begin{pmatrix} \rho[-\frac{h}{2}(c\partial_{xx} + |v|\partial_{yy}) + Q] & \rho[-\frac{h}{2}(c - |v|)\partial_{xy}] & \partial_x - \frac{h}{2}\frac{1}{c}Q_x \\ \rho[-\frac{h}{2}(c - |u|)\partial_{xy}] & \rho[-\frac{h}{2}(|u|\partial_{xx} + c\partial_{yy}) + Q] & \partial_y - \frac{h}{2}\frac{1}{c}Q_y \\ \rho(\partial_x - \frac{h}{2}\frac{u}{c}\partial_{xx}) & \rho(\partial_y - \frac{h}{2}\frac{v}{c}\partial_{yy}) & -\frac{h}{2}\frac{1}{c}\Delta + Q \end{pmatrix} \tag{53}$$

we can easily verified that FDA matrix is *factorizable*. The added multidimensional correction played a crucial role in achieving this property.

However, the FDA does not uniquely define a discrete scheme. Moreover, not all the discrete schemes corresponding to a certain FDA are *factorizable*. A *factorizable* scheme corresponding the above mentioned FDA was constructed on a quad-type grid. The details concerning the scheme as well as the Distributive Gauss-Seidel relaxation will be given elsewhere.

6 NUMERICAL EXPERIMENTS

6.1 Multidimensional upwinding

The purpose of the numerical experiments reported in this section is to demonstrate the robustness of the genuinely multidimensional upwind scheme and the quality of the numerical solutions obtained by its means. The multigrid algorithm employs the lexicographic Gauss-Seidel relaxation.
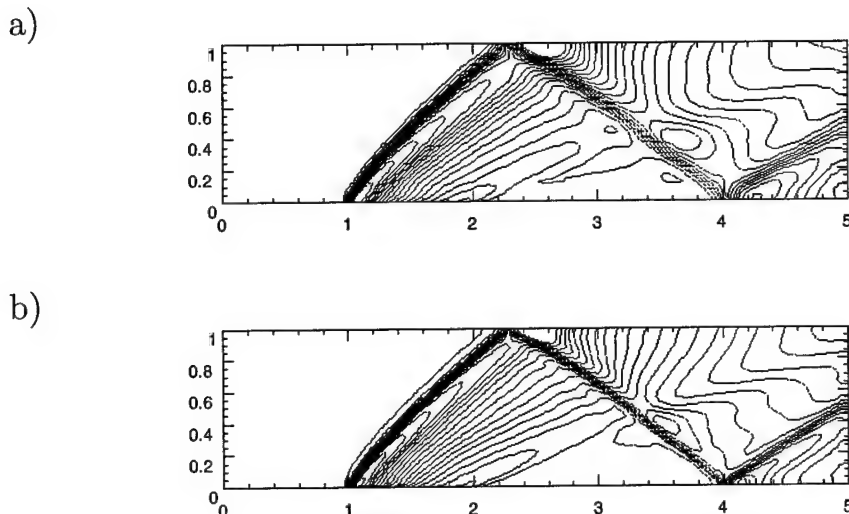
a)

b)

Fig. 3. Supersonic flow in a channel over a circular bump, grid 161×33: a) solution obtained by $2FMF - W(2,1)$ algorithm; b) the same, except that 3 more $W(2,1)$ cycles were performed on the finest grid.

It was demonstrated in [4],[5] that the $2FMG - W(2,1)$ algorithm employing the genuinely two-dimensional advection scheme (and Gauss-Seidel relaxation with direction-free ordering - like Red-Black) is capable of producing second order accurate (both in smooth regions and in terms of discontinuity location) solution to such a problem. We expect this to be true for the Euler equations as well (though more studies should be performed). Therefore, we present solutions obtained using this algorithm for a few testcases.

6.1.1 Supersonic flow in a channel with a bump.

The test case considered here is a supersonic (Mach=2.9) flow in channel with a circular bump. The bump is located at the lower wall of the channel at $1 \leq x \leq 2$ and its surface is a circular arch of $\pi/3$ and radius 1. Note that the actual shape of the domain is a rectangle. The influence of the bump on the flow is imposed through the boundary conditions: the velocity component normal to the surface of the bump at a certain location is being reflected.

The experiment uses the finest grid of the size 161×33 points. The solution obtained by $2FMG - W(2,1)$ algorithm is presented on Fig.3(a). Fig.3(b) presents the numerical solution obtained using the same algorithm but performing 3 more cycles (total 5) on the finest level.
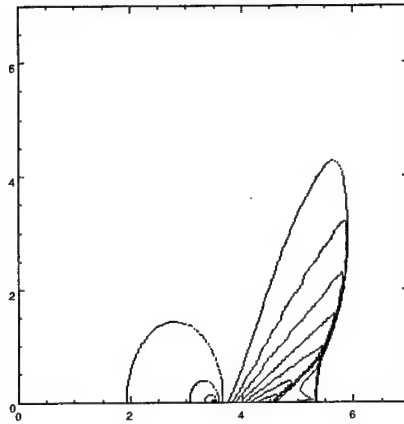
20

Fig. 4. Transonic flow over a wall with a circular bump: free stream Mach=.9, grid 200 × 200 pts.

6.1.2 Transonic flow over a circular bump

The testcase considered here is concerned with a transonic flow over a flat wall with a bump. The surface of the bump is again a circular arch of $\pi/3$ and radius 1 and its location is between $3.5 \leq x \leq 4.5$. Again, in order to keep the experiments simple at this stage of work, the bump is treated the same way as in the previous experiments. The free stream Mach=.9 in this case. The shock of the "fish-tail" shape is generated in this case (Fig.4).

6.1.3 Subcritical flow past an airfoil.

5 Here we present an experiment concerned with the subcritical flow past an airfoil. The testcase considered is Mach= .63 flow past $NACA$0012 airfoil at the angle of attack of 2°. The grid conatins about 9800 nodes. Pressure and density contours are presented by Figs.5 (a) and (b) respectively.

6.2 Pressure equation based schemes

Here we illustrate the efficiency of the multigrid algorithm that distinguishes between the different co-factors of the system. The testcase is a flow in a channel with a bump. The mutligrid cycle employed is $V(2, 1)$, the relaxation is lexicographic Gauss-Seidel in the flow direction. The solution contour plots for the incompressible case are presented on Fig.6. The sample convergence rates for the incompressible and compressible subsonic cases can be found in Fig.7. It can be obserevd that the residuals reduction per multigrid cycle is almost an order of magnitude. Also, the extensive set of computational results presented in [21] indicates that this behavior is essentially grid-independent.
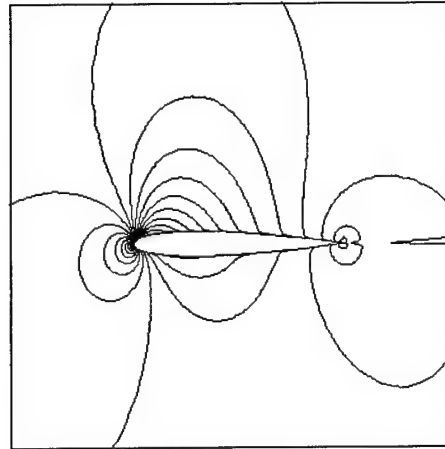
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a)
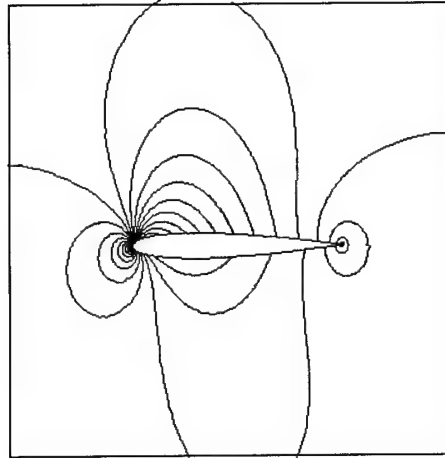
b)

Fig. 5. Subcritical flow (Mach = .63) over NACA0012 airfoil, angle of attack - 2°, grid of ≈ 9800 nodes: a) pressure contours; b) density contours.

7 CONCLUSIONS

An approach towards constructing a genuinely multidimensional upwind scheme was intoduced in [1–3]. The *fundamental* advantage of this approach for practical purposes is that the multidimensional high-resolution mechanism (unlike the standard one) does not damage the stability properties of the scheme. The conclusion made in this paper is that the genuinely multidimensional approach leads to a scheme that is also *factorizable*. The practical importance of this property is that an optimally efficient multigrid solver can be obtained through the construction of an appropriate Distributive Gauss-Seidel relaxation, that distinguishes between the different co-factors of the equations. Also, since the *factorizability* property can be easily verified, we suggest it is used as the definition of *genuine multidimensionality* of a scheme.
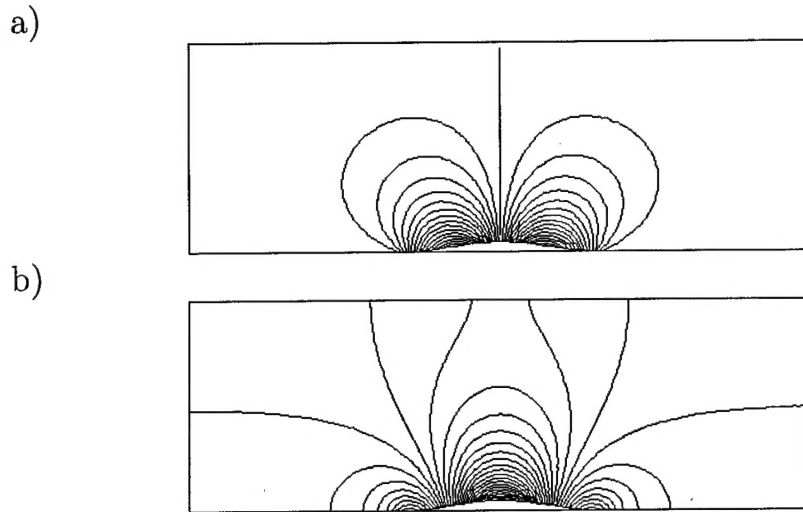
a)

b)

Fig. 6. Incompressible flow in a channel with a bump, grid of 97 × 33 vertices: a) v-velocity contours; b) pressure contours.
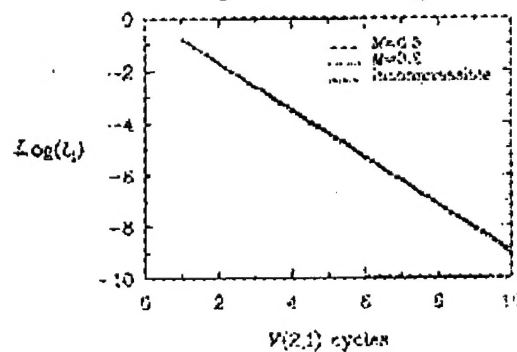

Fig. 7. Sample convergence histories for incompressible and compressible (Mach=0.2 and Mach=0.5) cases.

We also presented briefly the pressure equation based schemes and demonstrated on their example what efficiency of the multigrid solver can be expected if the distinction between the different co-factors of the equations is made by the algorithm.

Due to its generality, the genuinely multidimensional approach for discretization of the Euler equations may play a crucial role in constructing a general optimally efficient multigrid flow solver suitable for engineering computations. This is because

- it does not rely on casting the equations into any special form;
- extends to the compressible Navier-Stokes equations.

Acknowledgements

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